Collective Dynamics

Numerical and Theoretical Analysis of a New One-dimensional Cellular Automaton Model for Bidirectional Flows

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Abstract In recent years, research on mathematical models describing crowd dynamics has become increasingly important. Among this research, a two-dimensional mathematical model with the effect of body rotation describing bidirectional flows has been constructed, and its fundamental diagram has been shown to be qualitatively consistent with real experimental data from the perspective of flow rate inversion. However, this property has not been mentioned in one-dimensional mathematical models. In this paper, we introduce a new, simpler, one-dimensional cellular automaton model to focus on the direction of particles and the effects of mutual anticipation and flipping instead of body rotation by extending the well-known TASEP as a solvable lattice model. Our model was found to be qualitatively consistent with the actual phenomenon of flow rate inversion, both numerically and theoretically.

Keywords Pedestrian dynamics · bidirectional flows · cellular automaton

1 Introduction

In recent years, research on mathematical models describing crowd dynamics has become increasingly important. On the other hand, various findings have been obtained not only from mathematical models but also from real experiments [1]. For instance, experiments on unidirectional and bidirectional flow in crowds showed that the flow rate in unidirectional flow is higher than that in bidirectional flow in a certain density region. However, beyond the density region, a reversal occurred in which the bidirectional flow exhibited

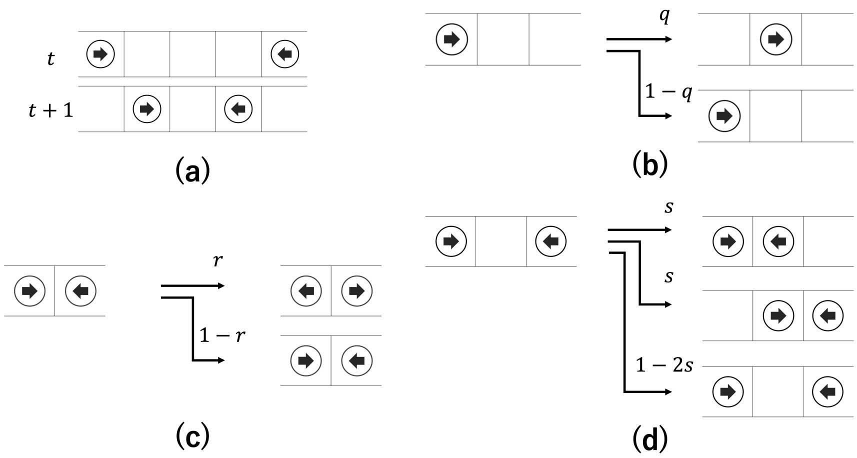


Figure 1 Illustration of each probability. (a) The particles remain the same from the initial direction. (b) Essentially, each particle hops with probability q. (c) If they are facing each other, two particles would swap places with probability r. (d) If the space between two particles facing each other is empty, one will hop with probability s.

a higher flow rate compared to the unidirectional flow [2]. While there are some one-dimensional cellular automaton models [3, 4] that represent bidirectional flows, none of them analytically show the reversal of flow. In this paper, the inversion in the flow rate is explained using a simpler one-dimensional cellular automaton model incorporating effects such as body rotation and mutual anticipation known from experiments. Our simpler model contributes to the understanding of flow rate inversion, both numerically and analytically.

The remainder of this paper is organized as follows. The next section presents a pedestrian model described as a partial difference equation, which corresponds to a cellular automaton in a special case. In addition, we compared the numerical and analytical results of the model and revealed an inversion in the flow rate. Finally, conclusions are presented in Section 3.

2 New mathematical model

2.1 Model

In this paper, we first introduce a new simpler one-dimensional cellular automaton model by focusing on the direction of particles and the effect of flipping instead of body rotation by extending the well-known Totally Asymmetric Simple Exclusion Process (TASEP) as a solvable lattice model. As shown in Fig. 1, each particle constantly moves to either side (Fig. 1(a)) and hops to the cell in front with probability $q \in [0, 1]$ (Fig. 1(b)) if the cell in front is vacant. The probability-determining directions cause two patterns that cannot occur in ASEP: First, if one particle is facing a neighboring particle, the particles flip

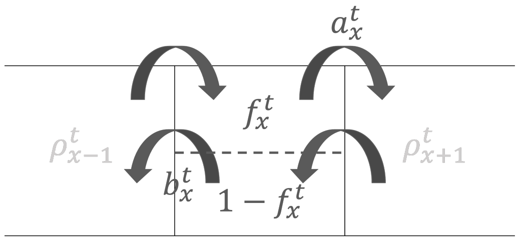


Figure 2 Conceptual diagram of our model.

with probability r (Fig. 1(c)). Second, if the neighboring cell in the direction of the hop is empty but there is a particle one cell further ahead facing it, one of the particles hops with probability s (Fig. 1(d)); this is based on an analogy with the introduction of a friction parameter [5]. In this study, we set $r = q^2 \in [0,1], s = q(1-q) \in [0, 0.25]$. Note that the setting $r = q^2$ signifies that when two pedestrians are adjacent, facing each other, and both attempt to move forward, they can do so successfully. More simply, it indicates that they rotate their bodies (as mentioned in [6]) to avoid a collision and proceed. On the contrary, the setting $r = q(1-q)$ represents the scenario where pedestrians facing each other avoid a collision through mutual anticipation [7]. This model is an extension of TASEP, and the appearance of flips is consistent with the behavior of other physical models [3, 8].

We then represent this model using the equation shown in Fig. 2, where the particle ρ_x^t in a cell moves with transition rates a_x^t and b_x^t; if $\rho_x^t = 1$, there is a particle in the cell, and if $\rho_x^t = 0$, there is no particle. As the conservation law holds for particles moving between cells, the following equation holds:

$$\rho_x^{t+1} = \rho_x^t - f_x^t \rho_x^t a_x^t - (1-f_x^t)\rho_x^t b_x^t + f_{x-1}^t \rho_{x-1}^t a_{x-1}^t + (1-f_{x+1}^t)\rho_{x+1}^t b_{x+1}^t \quad (1)$$

where f_x^t is a variable representing direction; if it takes the value 1, it represents a particle facing rightward, and if it takes the value 0, it represents a particle facing leftward. Because the direction is always constant at the initial value, for f_x^t, which represents direction, the following equation can be obtained:

$$f_x^{t+1} = \frac{f_x^t \rho_x^t (1-a_x^t) + f_{x-1}^t \rho_{x-1}^t a_{x-1}^t}{\rho_x^{t+1}}. \quad (2)$$

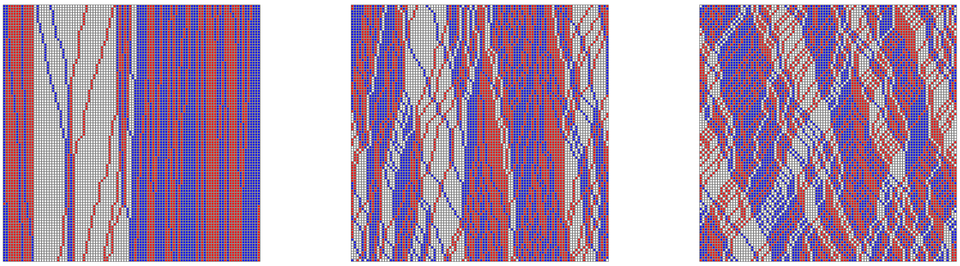
By rearranging equation (Eq. 2) and setting $r_x^t = f_x^t \rho_x^t$, we obtain the following equation:

$$r_x^{t+1} = r_x^t (1-a_x^t) + r_{x-1}^t a_{x-1}^t. \quad (3)$$

In addition, the transition rates a_x^t and b_x^t are as follows:

$$\begin{aligned} a_x^t = c_x^t [&(1-\rho_{x+1}^t)\{1-(\rho_{x+2}^t - r_{x+2}^t)\} \\ &+ (1-\rho_{x+1}^t)(\rho_{x+2}^t - r_{x+2}^t)(1-c_{x+2}^t), + c_{x+1}^t(\rho_{x+1}^t - r_{x+1}^t)] \quad (4) \\ b_x^t = c_x^t &\left[(1-\rho_{x-1}^t)(1-r_{x-2}^t) + \rho_{x-2}^t(1-\rho_{x-1}^t)(1-c_{x-2}^t) + c_{x-1}^t r_{x-1}^t\right]. \quad (5) \end{aligned}$$

c_x^t is a random variable that determines whether a particle in a cell at (x,t) moves forward; it takes value 1 with probability q; otherwise, it takes value 0. The first term in



(a) (b) (c)

Figure 3 Simulation results of our model. All boundary conditions are periodic boundary conditions, and the last 100 steps are taken from the 20,000 steps calculated. (a),(b), and (c) show the results for $q = 0.2, 0.5$, and 0.8, respectively.

parentheses on the right-hand sides of equations (Eq. 4) and (Eq. 5) represents the situation in Fig. 1(b). The second term represents the situation in Fig. 1(d), and the third term represents the situation in Fig. 1(c). In the present model, we set the variable $\rho_x^t \in \{0,1\}$ included in equations (Eq. 1), (Eq. 3), (Eq. 4), and (Eq. 5), but if we set $\rho_x^t \in [0,1]$, these equations still hold.

2.2 Results

First, we performed numerical simulations to verify the solution. Specifically, simulations were performed with the initial values $\rho_x^0 = 1$ and $r_x^0 = 1$ for $x \in \{0,1,\ldots,34\}$, $\rho_x^0 = 0$ and $r_x^0 = 1$ for $x \in \{65,66,\ldots,99\}$, while $\rho_x^0 = 0$ and $r_x^0 = 0$ were set for the other spaces. In Fig. 3, the model parameters were set to (a) $q = 0.2$, (b) $q = 0.5$, and (c) $q = 0.8$. In addition, the boundary condition is periodic, and parallel updates are used for updating. As shown in Fig. 3, the larger the value of the parameter q, the smaller the queue and the more it is moving overall. To illustrate this, we consider a diagram called the fundamental diagram, which considers the density on the horizontal axis and the flow rate on the vertical axis. The purpose of our proposed model is to present analytically determining the flow rate and show the flow reversal phenomena. Therefore, to obtain the fundamental diagram theoretically, we consider the situations for four cells. For instance, consider the transition probability matrix when the number of right-facing particles $N_R = 1$ and number of left-facing particles $N_L = 1$. The transition probability matrix representing the state transitions is as follows:

$$A = \begin{pmatrix} P1 & 0 & 0 & 0 & 0 & 0 & P2 & 0 & P4 & 0 & 0 & P4 \\ 0 & P1 & 0 & 0 & 0 & 0 & 0 & P2 & P4 & P4 & 0 & 0 \\ 0 & 0 & p1 & 0 & P2 & 0 & 0 & 0 & 0 & P4 & P4 & 0 \\ 0 & 0 & 0 & P1 & 0 & P2 & 0 & 0 & 0 & 0 & P4 & P4 \\ P2 & 0 & 0 & 0 & P3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P2 & 0 & 0 & 0 & P3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & P2 & 0 & 0 & 0 & P3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P2 & 0 & 0 & 0 & P3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & P4 & P4 & P5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P4 & 0 & 0 & P4 & 0 & P5 & 0 & 0 \\ 0 & 0 & 0 & 0 & P4 & P4 & 0 & 0 & 0 & 0 & P5 & 0 \\ 0 & 0 & 0 & 0 & 0 & P4 & P4 & 0 & 0 & 0 & 0 & P5 \end{pmatrix} \tag{6}$$

where $P1 = 1-q^2, P2 = q^2, P3 = (1-q)^2, P4 = q(1-q)$, and $P5 = 1-2q(1-q)$. Applying the Perron–Frobenius theorem to the irreducible transition probability matrix A, it can be shown that the eigenvalue of the largest absolute value is 1, which is a simple eigenvalue. Thus, the Markov chain defined by this matrix has a unique stationary distribution. Therefore, the eigenvector corresponding to eigenvalue 1 is a constant multiple of the stationary distribution. Then, the eigenvector is obtained as follows:

$$\left(\frac{2-q}{q}, \frac{2-q}{q}, \frac{2-q}{q}, \frac{2-q}{q}, 1,1,1,1,1,1,1,1\right). \tag{7}$$

Let us denote the absence of particles in a cell as 0, the presence of right-facing particles as R, and the presence of left-facing particles as L. Then, the stationary distribution can be expressed as follows:

$$P(00RL) = P(0RL0) = P(RL00) = P(L00R) = C\frac{2-q}{q}, \tag{8}$$

$$P(00LR) = P(0LR0) = P(LR00) = P(R00L) = C, \tag{9}$$

$$P(0R0L) = P(R0L0) = P(0L0R) = P(L0R0) = C, \tag{10}$$

where C is the normalization constant. The flow rate Q can be calculated based on the stationary distribution obtained here. Specifically, it is expressed as follows:

$$Q = \frac{(\text{\# particles moved}) \times (\text{Transition probability}) \times (\text{Stationary distribution})}{(\text{Number of cells})}. \tag{11}$$

To calculate the flow rate Q specifically for the case when $N_R = 1$ and $N_L = 1$, the normalization constant C is determined that

$$\begin{aligned} C &= \frac{1}{4 \times \frac{2-q}{q} + 8 \times 1} \\ &= \frac{q}{8+4q}. \end{aligned} \tag{12}$$

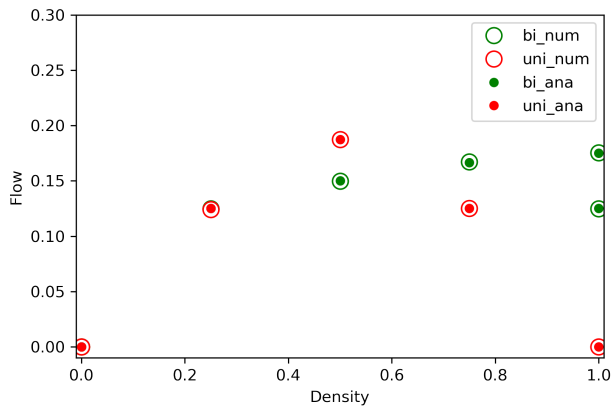


Figure 4 Fundamental diagrams. The red dots represent the fundamental diagram of TASEP, and the green dots represent the fundamental diagram of our model. Large circles represent numerical results, and dots represent theoretical results. It can be seen that the numerical and theoretical results are in agreement, and moreover, they represent the flow reversal seen in real phenomena.

When substituting equation (Eq. 12) into equations (Eq. 8), (Eq. 9), and (Eq. 10), stationary distribution can be determined. When $N_R = 1$ and $N_L = 1$, the flow rate is denoted as $Q_{1,1}$, and it is determined that

$$\begin{aligned} Q_{1,1} &= \left(2\times q^2 \times \frac{q}{8+4q} \times \frac{2-q}{q} \times \frac{1}{4}\right) \times 4 + \left(2\times q^2 \times \frac{q}{8+4q} \times 1 \times \frac{1}{4}\right) \times 4 \\ &\quad + \left(1 \times 2q(1-q) \times \frac{q}{8+4q} \times 1 \times \frac{1}{4}\right) \times 4 + 2 \times \left(1 \times q(1-q) \times \frac{q}{8+4q} \times 1 \times \frac{1}{4}\right) \times 4 \\ &= \frac{q^2(2-q)}{2+q}. \end{aligned} \tag{13}$$

These calculations were performed from $N = 0$ to $N = 4$, and the results compared through simulations are shown in Fig. 4. The green and red dots represent bidirectional and unidirectional flow, respectively. Fig. 4 shows that some particles can move in bidirectional flows when the density $\rho = 1.0$. Such behavior arises because the particles facing each other can move forward with probability $r = q^2$. This result is natural in the model setting. This result is natural in the model setting. In addition, there are two flow rates when the density is 1.0; these two points at $\rho = 1.0$ are because the flow rate varies with the number of people coming from the left and right. In a bidirectional flow, the model illustrates that the flow rate fluctuates based on the disparity in the number of people coming from each direction. More precisely, the flow rate is higher when the ratio of rightward to leftward particles is equal ($N_L = N_R$), and it is lower when the ratio is skewed ($N_L > N_R$ or $N_L < N_R$). Fig. 4 shows that the actual theoretical and simulation results, which can describe the properties of flow inversion, as observed in the experiment [2].

3 Conclusion

By extending TASEP to introduce particle direction and flips, we constructed a new one-dimensional model that represents both directions in the human flow. Specifically, we made particles keep moving in one of the directions and defined how they would behave with probability in the event of a collision. Then, we constructively introduced the equations that represent such a model. In addition, we obtained the exact flow rates at some finite number of cells to obtain an eigenvector corresponding to the eigenvalue 1. We then showed the behavior of the times at which the flow rate was obtained by simulation. Future studies will aim to determine the exact flow rate for an arbitrary number of cells.

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Author Contributions Kazuya Okamoto: Conceptualization, Methodology, Writing – original draft / Akiyasu Tomoeda: Writing – review and editing, Supervision

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