Collective Dynamics

Exploring the Braess Paradox: Static Versus Dynamic Assignment

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Abstract The Braess paradox is a well-known phenomenon initially observed in road traffic flow. It points out that increasing network capacity can lead to poorer performance in congested situations, when the drivers attempt to optimise their travel time individually. This paradox is not limited to road transport, but also extends to various information networks. In this article, we examine the Braess paradox in a closed network where demand remains constant. First, we determine the user and global optima of the deterministic system in stationary states with flow balancing. We present explicit formulae for the density intervals at which the Braess paradox occurs. We then compute Monte Carlo simulations of a stochastic mesoscopic traffic model using aggregate data obtained from a queueing model to explore the results. Static assignment models match the deterministic stationary results. In addition, the simulations assess the effectiveness of dynamic assignment, whereby drivers select routes in real time to minimise travel time. Interestingly, behaviour with dynamic assignment deviates from the generic static assignment results, particularly in highly congested situations. These results emphasise the significance of dynamic route selection in relation to Braess's paradox.

Keywords Braess paradox · closed system · flow balancing · user & system optima · interacting particle systems · dynamic assignment · Monte Carlo simulation

1 Introduction

The Braess paradox was revealed in 1920 in the seminal work of Arthur Pigou [1] and was mathematically formulated in 1968 by Dietrich Braess [2]. The paradox points out that in congested traffic, increasing network capacity by introducing additional roads can make the situation worse if the users optimise their travel times individually. Essentially, the paradox hinges on the distinction between the user (or Nash) equilibrium and the system equilibrium, as formalised in traffic engineering by Wardrop's principles [3]. Although the paradox has been highlighted and studied for over 60 years, designing networks for selfish agents is still challenging today [4–8].

The Downs-Thomson paradox, also known as *Paradox of Traffic*, is related to Braess paradox. It describes the situation where new additional roads lead to more traffic jams in the long term. This is related to the fact that new roads lead to an increased traffic demand. In contrast, Braess paradox assumes a constant demand.

Initially, the paradox concerns road traffic networks [9] and its prevalence in random networks [10, 11]. The Braess paradox in traffic networks has been frequently documented in practice, for example in Seoul (South Korea) [12,13], Stuttgart (Germany) [14] or Manhattan (United States) [15]. The paradox has also been observed in many other areas, such as communication networks [16, 17], electrical and hydraulic networks [18–24], biological organisms [25–27], chemical reaction networks [28, 29], mechanical networks (spring networks) [30, 31], or quantum transport [32–34]. See also [35] from D. Braess homepage listing research articles on the paradox until 2019 and [36] for a systematic bibliographic review covering the period 1968–2022. The Braess paradox is also analysed through the prism of game theory, where it is generally considered as a non-cooperative multiplayer game [37–39], and is even found in basketball games [40]. Braess paradox is multidisciplinary and seems to be more and more topical. Fig. 1 shows the number of research articles per year on Braess paradox, traffic flow, game theory and communication networks. The trends are clearly upward.

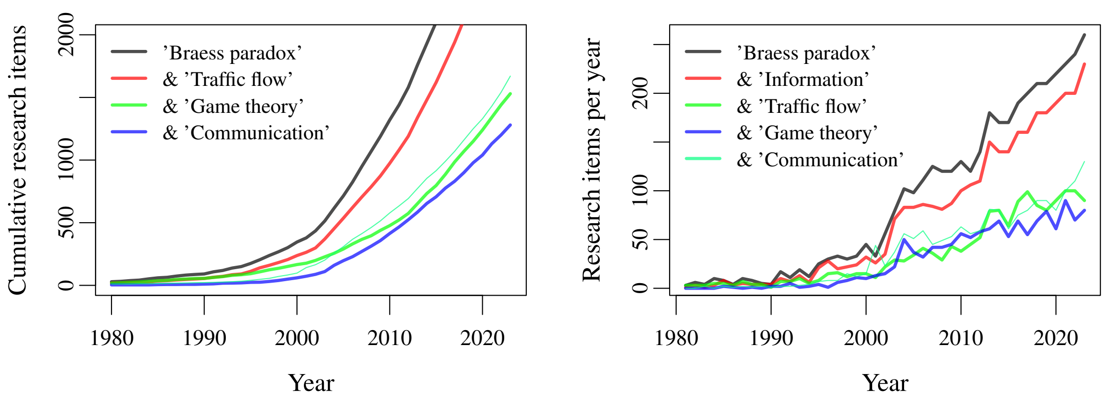

Figure 1 Number of research articles from 1980 to 2023 with the tags 'Braess paradox', 'Traffic flow', 'Game theory', and 'Commmunication' in the Google Scholar engine. Left panel: cumulative number of research articles. Right panel: number of articles per year.

1.1 Braess paradox

Consider a network with an origin O, a destination D, and two internal nodes A and B. Initially, only two routes are possible: crossing through A (route OAD) or crossing through B (route OBD), see Fig. 2. The route OAD begins on an urban road c_A, where the travel time is proportional to the number of users and is characterized by a time headway coefficient ω. This coefficient represents the time it takes for a vehicle to reach the position of the vehicle in front, assuming a constant speed. The route then becomes a highway h_A, which has a large capacity and a constant travel time T_h.

The route OBD is the opposite, starting from the highway h_B before the city centre c_B. In the example shown in Fig. 2, there are $N = 10$ vehicles, $\omega = 1$ and $T_h = 10$. The user optimum distributes equally $n = 5$ vehicles on each of the two routes and, assuming that the drivers travel together in a static framework, the travel time from the origin to the destination is $\omega n + T_h = 5 + 10 = 15$ for all the vehicles. Now consider a new route, the shortcut, which immediately connects A to B instantaneously and is used by $m \in [0,N]$ vehicles. This is an artificial concept since the travel time is zero and therefore, theoretically, the length of this new route is zero. In fact, all the vehicles have individually interest to switch to the route $OABD$ composed of the roads c_A and c_B so that, in user equilibrium, $n = 0$, $m = 10$ and, assuming again that the drivers all travel together, the travel time is $\omega m + \omega m = 10 + 10 = 20$ for all the vehicles. This is the Braess paradox: we increase the capacity of the network and, for the same demand, the user's travel time increases.

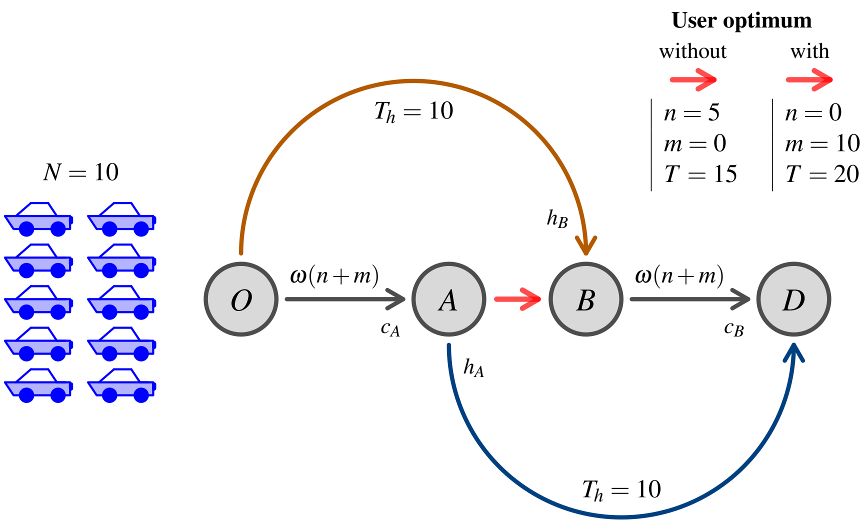

Figure 2 Illustrative example of the Braess Paradox with $N = 10$ vehicles on a network with origin O and destination D. There are three possible routes: OAD, OBD and $OABD$. The parameter $m \in [0,N]$ is the number of vehicles taking the route $OABD$ with the shortcut from A to B. The roads c_A (from O to A) and c_B (from B to D) are city centres where the travel time is proportional to the demand (headway coefficient $\omega = 1$), while the roads h_A (from A to D) and h_B (from O to B) are highways with large capacity and constant travel time $T_h = 10$. The user's travel time is 15 without the route $OABD$, while it increases to 20 with the shortcut.

1.2 Objectives and organisation of the manuscript

Although the static illustration described in the previous section can be instructive, it does not take into account complex dynamical aspects of road traffic flow. In fact, the illustration shown in Fig. 2 is a static depiction of how vehicles move on the road. Most current studies on the Braess paradox are time-dependent analyses based on dynamic traffic models. The Braess paradox consists of coupling the traffic model with an optimal control strategy. For open systems, traffic demand is controlled by entry and exit rates, and the number of vehicles in the system can fluctuate. For closed systems, the number of vehicles is conserved and traffic demand is constant at any time. Macroscopic network traffic models depend on demand and supply functions on edges and nodes [41–43] of the network, while macroscopic traffic flow models are based on nonlinear partial differential for the flow, the density and the mean speed [44]. Mesoscopic traffic models rely on kinetic equations for the probability density function of the vehicle distribution [18, 24, 45, 46], or queuing models [47–50]. Microscopic approaches model the motion of each vehicle and allow to include individual route choice models, but these can quickly prove computationally intensive [51–54]. More recent models take into account for real-time traffic state information, dynamic assignment and stochastic aspects [6, 55, 56].

In this article, we examine the Braess paradox in a closed system. The traffic demand remains constant and we also assume certain symmetry properties. First, we use the deterministic system in stationary states with flow balancing to determine the density ranges at which the Braess paradox arises. We systematically compare the user optimum, where the Braess paradox occurs, and the system optimum. Next, we present Monte Carlo simulations of a stochastic interacting particle system through a continuous-time Markovian process. In contrast to microscopic models which are computationally intensive, we use a mesoscopic approach using queueing process, which allows us to consider individual dynamics with computational requirements close to those of macroscopic models. We first simulate two different static route choice models, which systematically yield average estimates that align with the deterministic theoretical results. Additionally, we numerically investigate two dynamic assignment models, where drivers choose their routes in real time according to the current state of the network, similar to a navigator, and a minority game where the drivers choose the route based on their past travel times. Interestingly, the dynamic assignment results show some deviations from the generic results obtained with static assignment.

The article is organised as follows. The notation and modelling assumptions are given in the next subsection. The mathematical properties of the deterministic symmetric systems with user and system optimums in stationary states are given in Sec. 2, while the Monte Carlo simulation results of the stochastic interacting particle system with various static and dynamic assignment models are presented in Sec. 3. Some discussion and concluding remarks are reported in Sec. 4.

1.3 Notations and modelling assumptions

We consider a closed-loop system with periodic boundaries containing $N \geq 1$ vehicles, where a vehicle reaching the destination is returned to the origin. The total number of vehicles N remains constant. We denote $n_A \in [0,N]$ the number of vehicles using the route *OAD*, $n_B \in [0,N]$ the number of vehicles using the route *OBD*, and $m \in [0,N]$ the number of vehicles using the route *OABD*, so that $n_A + n_B + m = N$. For simplicity, we consider in the following the symmetric case, where $n_A = n_B =: n$ and

$$2n + m = N. \tag{1}$$

In addition to the total number of vehicles N, the closed system has only one variable: the number of vehicles taking the shortcut $m \in [0,N]$.

The route *OAD* (respectively *OBD*) is composed of the roads c_A and h_A (respectively c_B and h_B), h_A and h_B being highways with constant travel time $T_h > 0$, and c_A and c_b city roads, where the travel time is proportional to the number of users with a headway coefficient $\omega > 0$. The route *OABD* including the shortcut between the cities is composed of the roads c_A and c_B (see Fig. 2). We denote n_{h_A}, n_{h_B}, n_{c_A}, and $n_{c_B} \in [0,N]$ the number of vehicles on each of the roads and suppose here again the symmetry $n_{h_A} = n_{h_B} =: n_h(m)$ and $n_{c_A} = n_{c_B} =: n_c(m)$. The vehicles conservation in the closed system gives

$$2(n_c(m) + n_h(m)) = N, \qquad m \in [0,N]. \tag{2}$$

Fig. 3 provides a summary of the notation used, the assumptions made for simplification about the symmetry, and the vehicle conservation.

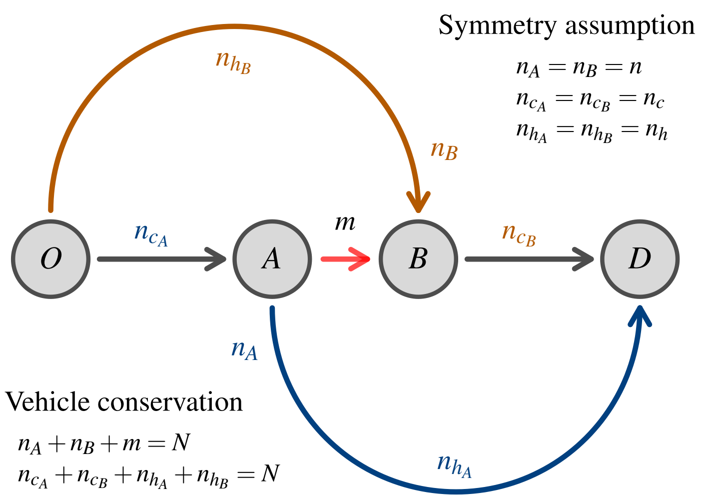

Figure 3 Summary of the notation for the vehicle numbers on the different routes of the system, symmetry assumptions and vehicle conservation.

Definition 1.1 (Travel time on the routes with highway)**.** The travel time $T_{\smile}(m)$ for the routes *OAD* and *OBD* including the city road and the highway is given by

$$T_{\smile}(m) = \omega n_c(m) + T_h, \qquad m \in [0,N]. \tag{3}$$

Definition 1.2 (Travel time on the route with the shortcut)**.** The travel time $T_{-}(m)$ for the route *OABD* using the shortcut consisting of the two city roads is

$$T_{-}(m) = 2\omega n_c(m), \qquad m \in [0,N]. \tag{4}$$

Proposition 1.1 (Mean travel time)**.** *The mean travel time of the vehicles $T(m)$ from origin O to destination D according to the number of vehicles m using the shortcut is given by*

$$T(m) = \omega n_c(m)\frac{N+m}{N} + T_h\frac{N-m}{N}, \qquad m \in [0,N]. \tag{5}$$

Proof. In fact, we have for all $m \in [0,N]$

$$\begin{aligned} T(m) &= \frac{1}{N}\big((N-m)T_{\frown}(m) + mT_{-}(m)\big) \\ &= \frac{1}{N}\big((N-m)\big(T_h + \omega n_c(m)\big) + 2m\omega n_c(m)\big) \\ &= \omega n_c(m)\frac{N+m}{N} + T_h\frac{N-m}{N}. \end{aligned} \tag{6}$$

□

Remark 1. *The vehicles mean travel time is equal to*

$$T(0) = T_h + \omega n_c(0) \tag{7}$$

if no vehicles use the shortcut OABD, while it is equal to

$$T(N) = 2\omega n_c(N) \tag{8}$$

if all the vehicles use the shortcut.

To simplify the notations, we also use the travel time relative to travel time on the highway T_h

$$\mathscr{T} = \frac{T}{T_h} \geq 0, \tag{9}$$

the proportion of vehicles using the shortcut

$$\mu = \frac{m}{N} \in [0,1], \tag{10}$$

and the control parameter

$$\alpha = \frac{N\omega}{T_h} \geq 0. \tag{11}$$

The parameter α primarily quantifies the density level in the system, assuming that the time headway ω and travel time on the highway T_h are constant. The higher the value of α, the higher the density.

2 Behaviour in stationary states

In the following, we consider a stationary distribution of the vehicles resulting from a flow balance equation in a deterministic framework. In fact, the number of vehicles on each of the roads in stationary states is such that the incoming flow is equal to the outgoing flow (see Fig. 4). For that end, we introduce $k \in [0, n_c]$, the number of vehicles on each city road that do not use the shortcut, where n_c is the total number of vehicles on each city road in stationary states.

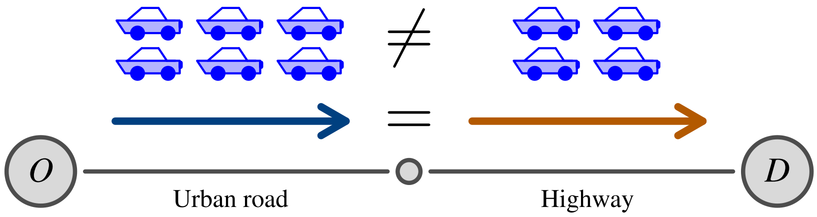

Figure 4 Scheme for the stationary state. The vehicle speeds and travel times can differ between the two roads. However, the densities are such that the traffic volumes are the same.

Proposition 2.1 (Mean travel time in stationary states)**.** *We have in stationary state with flow balancing for some $k \in [0, n_c]$*

$$T_{\sim}(k,m) = \omega\left(k + \frac{m}{2}\right) + T_h, \qquad m \in [0,N], \tag{12}$$

$$T_{-}(k,m) = \omega(2k+m), \qquad m \in [0,N], \tag{13}$$

and

$$T(k,m) = \omega\left(k + \frac{m}{2}\right)\frac{N+m}{N} + T_h\frac{N-m}{N}, \qquad m \in [0,N]. \tag{14}$$

or again, with the proportion of vehicle using the shortcut and relative travel times

$$\mathscr{T}_{\sim}(k,\mu) = 1 + \alpha\left(\frac{k}{N} + \frac{\mu}{2}\right), \qquad \mu \in [0,1], \tag{15}$$

$$\mathscr{T}_{-}(k,\mu) = \alpha\left(\frac{2k}{N} + \mu\right), \qquad \mu \in [0,1], \tag{16}$$

and

$$\mathscr{T}(k,\mu) = \alpha\left(\frac{k}{N} + \frac{\mu}{2}\right)(1+\mu) + 1 - \mu, \qquad \mu \in [0,1]. \tag{17}$$

Proof. We have by definition thanks to the symmetry of the two routes with no shortcut that $m = 2(n_c - k)$, i.e.,

$$n_c(k,m) = k + \frac{m}{2}, \qquad m \in [0,N]. \tag{18}$$

Then, using (3), (4), and (5), the travel times are

$$\begin{aligned} T_{\smile}(k,m) &= \omega n_c(k,m) + T_h & (19)\\ &= \omega\left(k+\frac{m}{2}\right) + T_h, \qquad m \in [0,N], & (20)\end{aligned}$$

$$\begin{aligned} T_{-}(k,m) &= 2\omega n_c(k,m) & (21)\\ &= \omega(2k+m), \qquad m \in [0,N], & (22)\end{aligned}$$

and

$$\begin{aligned} T(k,m) &= \omega n_c(m)\frac{N+m}{N} + T_h\frac{N-m}{N} & (23)\\ &= \omega\left(k+\frac{m}{2}\right)\frac{N+m}{N} + T_h\frac{N-m}{N}, \qquad m \in [0,N], & (24)\end{aligned}$$

Finally the relative travel times (15), (16) and (17) are recovered by using $\mathscr{T} = T/T_h$, $\mu = m/N$ and $\alpha = N\omega/T_h$. □

Remark 2. *We have by definition that*

$$k(m) = n_c(m) - \frac{m}{2} \leq n_c(m), \qquad m \in [0,N]. \tag{25}$$

Remark 3. *We also have that*

$$n_h(k,m) = \frac{N-m}{2} - k \tag{26}$$

on the highways and, the vehicle conservation holds in the symmetric system as

$$2(n_h(k,m) + n_c(k,m)) = 2\left(k + \frac{m}{2} + \frac{N-m}{2} - k\right) = N \tag{27}$$

holds for any k and $m \in [0,N]$.

Remark 4. *The travel times (A1) and (A2) in uniform states are recovered for*

$$k = \frac{n}{2} = \frac{(N-m)}{4}, \qquad m \in [0,N], \tag{28}$$

see in appendix.

Proof. In fact, we have for $k = (N-m)/4$ in (12)

$$\begin{aligned} T_{\smile}\left(\frac{N-m}{4}, m\right) &= \omega\left(\frac{N-m}{4} + \frac{m}{2}\right) + T_h\\ &= \omega\frac{N+m}{4} + T_h\\ &= T_{\smile}^{u}(m), \qquad m \in [0,N], \end{aligned} \tag{29}$$

while using (13) we obtain

$$\begin{aligned} T_-\Big(\frac{N-m}{4},m\Big) &= \omega\Big(\frac{N-m}{2}+m\Big) \\ &= \omega\frac{N+m}{2} \\ &= T_-^u(m), \qquad m\in[0,N]. \end{aligned} \tag{30}$$

The same holds for the relative travel times using $\mathscr{T}=T/T_h$, $\mu=m/N$ and $\alpha=N\omega/T_h$. □

We need two more equations to fix k and m. The first is the balance equation where the system is stationary, i.e., the inflows and outflows over the roads are equal.

Proposition 2.2 (Flow balancing)**.** *The flow balancing equation for the stationary system is given by*

$$\frac{1}{T_h}\Big(\frac{N-m}{2}-k\Big)=\frac{k}{\omega(k+m/2)}, \qquad m\in[0,N]. \tag{31}$$

Proof. The flow on the urban road of the vehicles that do not use the shortcut is $k/(\omega n_c(k,m))$ while the flow on the highway is $n_h(k,m)/T_h$. The flow are equal and the system is stationary if

$$\frac{1}{T_h}n_h(k,m)=\frac{k}{\omega n_c(k,m)}, \qquad m\in[0,N]. \tag{32}$$

Using (18) and (26), this is

$$\frac{1}{T_h}\Big(\frac{N-m}{2}-k\Big)=\frac{k}{\omega(k+m/2)}, \qquad m\in[0,N]. \tag{33}$$

□

The last equation closing the system is given by the user and system optimum, in addition to the reference system with no shortcut where $m=0$.

2.1 Reference system without shortcut

Proposition 2.3 (Mean travel time in stationary states without shortcut)**.** *The reference travel time without shortcut in stationary states is given by*

$$T_{ref}=\begin{cases} T_h, & 0\le N<2T_h/\omega, \\ \omega N/2, & N\ge 2T_h/\omega. \end{cases} \tag{34}$$

or again, using the relative travel time

$$\mathscr{T}_{ref}=\begin{cases} 1, & 0\le \alpha<2, \\ \alpha/2, & \alpha\ge 2. \end{cases} \tag{35}$$

Proof. With no shortcut (i.e., $m = 0$), the balance equation (31) gives

$$k_{\mathrm{ref}} = \begin{cases} 0 & 0 \le N < 2T_h/\omega, \\ N/2 - T_h/\omega & N \ge 2T_h/\omega. \end{cases} \tag{36}$$

Using (14), the corresponding travel time is then given by

$$T_{\mathrm{ref}} = T(k_{\mathrm{ref}}, 0) = \begin{cases} T_h, & 0 \le N < 2T_h/\omega, \\ \omega N/2, & N \ge 2T_h/\omega. \end{cases} \tag{37}$$

The relative reference travel time (35) is then recovered using $\mathscr{T} = T/T_h$ and $\alpha = N\omega/T_h$. $\square$

2.2 User optimum

Proposition 2.4 (User optimum in stationary states)**.** *The user optimum in stationary states is given by*

$$m_* = \begin{cases} N, \\ 4T_h/\omega - N, \\ 0, \end{cases} \qquad T_* = \begin{cases} \omega N, & 0 \le N < 2T_h/\omega, \\ 2T_h, & 2T_h/\omega \le N < 4T_h/\omega, \\ \omega N/2, & N \ge 4T_h/\omega, \end{cases} \tag{38}$$

or again, with the proportion of vehicle using the shortcut and relative travel time,

$$\mu_* = \begin{cases} 1, \\ 4/\alpha - 1, \\ 0, \end{cases} \qquad \mathscr{T}_* = \begin{cases} \alpha, & 0 \le \alpha < 2, \\ 2, & 2 \le \alpha < 4, \\ \alpha/2, & \alpha \ge 4. \end{cases} \tag{39}$$

Proof. In the user equilibrium, the travel times over the routes *OAD* and *OBD* and the route *OABD* with the shortcut are equal, i.e.,

$$T_{\frown}(\tilde{k}_*, \tilde{m}_*) = T_{-}(\tilde{k}_*, \tilde{m}_*). \tag{40}$$

Using (12) and (13), we obtain

$$T_h + \omega\left(\tilde{k}_* + \frac{\tilde{m}_*}{2}\right) = \omega\left(2\tilde{k}_* + \tilde{m}_*\right) \tag{41}$$

and

$$\tilde{k}_* = \frac{T_h}{\omega} - \frac{\tilde{m}_*}{2}. \tag{42}$$

Then, the balance equation (31) provides

$$\tilde{m}_* = \frac{4T_h}{\omega} - N. \tag{43}$$

Finally, bounding $\tilde{k}_*$ in $[0,(N-m_*)/2]$ and $\tilde{m}_*$ in $[0,N]$, we obtain

$$k_* = \begin{cases} 0, & 0 \leq N < 2T_h/\omega, \\ N/2 - T_h/\omega, & N \geq 2T_h/\omega, \end{cases} \tag{44}$$

$$m_* = \begin{cases} N, & 0 \leq N < 2T_h/\omega, \\ 4T_h/\omega - N, & 2T_h/\omega \leq N < 4T_h/\omega, \\ 0, & N \geq 4T_h/\omega, \end{cases} \tag{45}$$

and, using (14),

$$T_* = T(k_*,m_*) = \begin{cases} \omega N, & 0 \leq N < 2T_h/\omega, \\ 2T_h, & 2T_h/\omega \leq N < 4T_h/\omega, \\ \omega N/2, & N \geq 4T_h/\omega. \end{cases} \tag{46}$$

Finally the proportion of vehicles using the shortcut and relative travel times (39) are recovered by using $\mathscr{T} = T/T_h$, $\mu = m/N$ and $\alpha = N\omega/T_h$. □

Remark 5. *The travel time difference to the reference system with no shortcut reads*

$$\Delta T = T_* - T_{ref} = \begin{cases} \omega N - T_h \leq 0, & 0 \leq N < T_h/\omega, \\ \omega N - T_h \geq 0, & T_h/\omega \leq N < 2T_h/\omega, \\ 2T_h - \omega N/2 \geq 0, & 2T_h/\omega \leq N < 4T_h/\omega, \\ 0, & N \geq 4T_h/\omega. \end{cases} \tag{47}$$

The Braess paradox arises in stationary states for $T_h/\omega \leq N \leq 4T_h/\omega$, i.e., $1 \leq \alpha \leq 4$.

Remark 6. *The stationary states are uniform for the user optimum as $N \leq 4T_h/\omega$ (compare (38) and (A18) in appendix). In fact, the user optimum implies that travel times on the urban road and highway are equal. The balance equation (31) makes then equal the densities as well. There is no similar relationship for the reference system without shortcut, for the system equilibrium or for $N \geq 4T_h/\omega$, i.e., $\alpha \geq 4$, as no travel time equality is required anymore.*

2.3 System optimum

After simplifications, the balance equation (31) reduces to the quadratic equation

$$k^2 + k\Big(\frac{T_h}{\omega} - \frac{N}{2} + m\Big) - m\frac{N-m}{4} = 0, \qquad m \in [0,N], \tag{48}$$

whose biggest root is

$$k(m) = \frac{1}{2}\Big(\frac{N}{2} - m - \frac{T_h}{\omega} + \sqrt{\Delta(m)}\Big), \qquad m \in [0,N], \tag{49}$$

with

$$\Delta(m) = \left(\frac{N}{2} - m - \frac{T_h}{\omega}\right)^2 + m(N-m), \qquad m \in [0,N]. \tag{50}$$

The mean travel time (14) is given by

$$\begin{aligned} T(k(m),m) &= \omega\left(k(m) + \frac{m}{2}\right)\frac{N+m}{N} + T_h\frac{N-m}{N} \\ &= \omega\left(N + 2\sqrt{\Delta(m)}\right)\frac{N+m}{4N} + T_h\frac{N-3m}{2N}. \end{aligned} \tag{51}$$

Finally, the number of vehicles using the shortcut in the system optimum that minimises the mean travel time is the solution of

$$m_{\text{opt}} = \arg\min_{m\in[0,N]} T(k(m),m). \tag{52}$$

Such an optimum is solved numerically. The corresponding travel time is given with (14) by

$$T_{\text{opt}} = T(k(m_{\text{opt}}), m_{\text{opt}}) \tag{53}$$

The mean travel time and number of vehicles taking the shortcut for the reference system and the user and system optimums in stationary states are shown in Fig. 5. The latest is obtained numerically using R software package and the function `optimize` [57]. For $N = T_h/\omega$, i.e., for $\alpha = 1$, the system optimum abruptly switches from the user optimum to the reference without shortcut. The corresponding proportion of users choosing the shortcut switches from 1 to 0 with a transition phase in the vicinity of the critical setting $\alpha = 1$ (see Fig. 5, bottom panel). The Braess paradox (orange areas) occurs for the density range $T_h/\omega \leq N \leq 4T_h/\omega$, i.e., $1 \leq \alpha \leq 4$. The range more spread out than in uniform states. The paradox is also maximal for $N = 2T_h/\omega$, i.e., $\alpha = 2$, where the lost time is again more important. In addition, the use of the shortcut and its improvement in terms of performance for the system optimum is limited (compare the blue dashed curve and the red one in Fig. 5).

3 Stochastic simulation models

We consider a stochastic interacting particle system by a Markovian queuing model in continuous time, where the vehicle travel times over each of the roads are independent exponential random clocks. Unlike microscopic traffic approaches including a motion model for each vehicle [51–54], we aggregate the performance on each road using queues. This considerably reduces computational complexity [58]. However, the vehicles remain individually considered. In this sense, the approach is mesoscopic. The stochastic approach with exponential clocks can be simulated event-driven in continuous time, without the need of a numerical discretisation scheme. A comparable event-based simulation process is used in [59] in a deterministic framework.

The network is composed of four sites, the two highways and the two urban roads. Each site contains a certain number of vehicles. The jump rate of a vehicle from a site

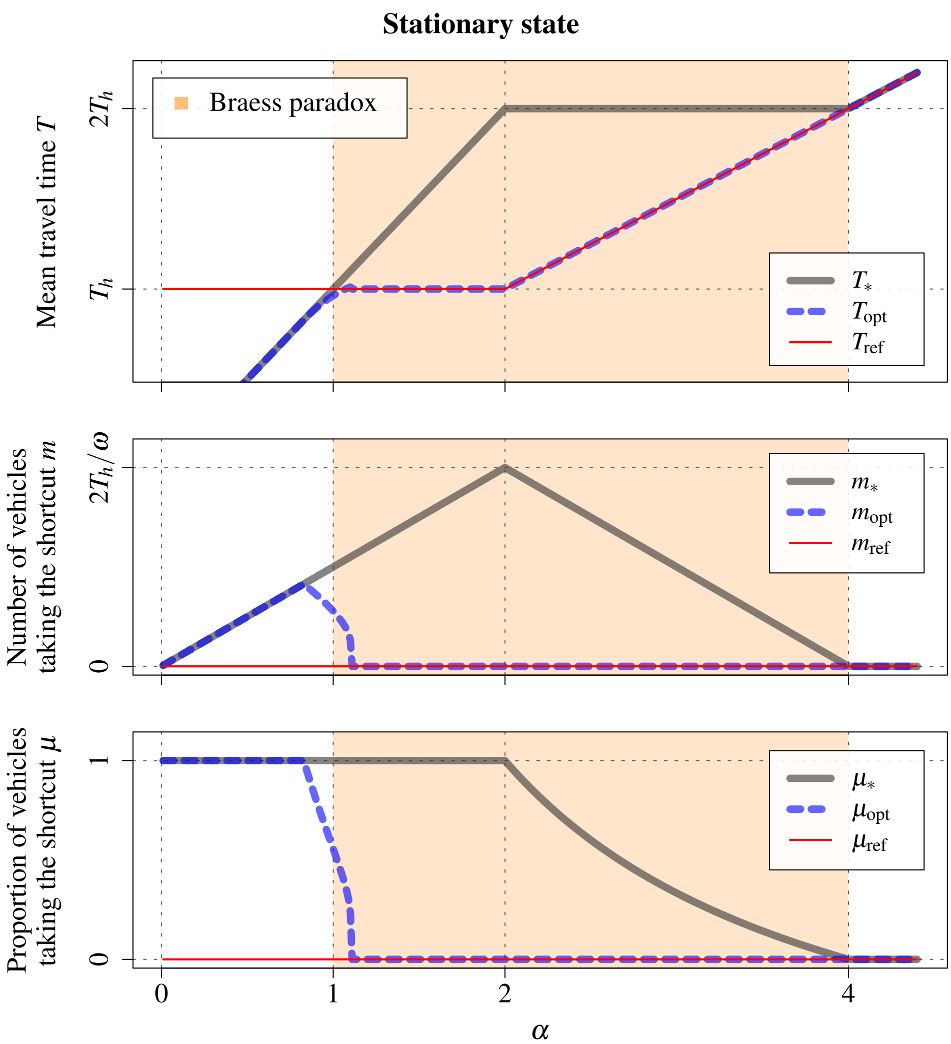

Figure 5 Mean travel time (upper panel) and number and proportion of drivers choosing the shortcut (lower panels) in stationary states according to $\alpha = N\omega/T_h$ for the user optimum (38) and (39) – grey curve, the system optimum (52) and (53) – blue dashed curve optimums, and the reference system without the shortcut (34) – red curve. The orange areas indicate the density range for which the Braess paradox occurs.

to the next one depends on the number of vehicles on the departure site. In average, the travel times are equal to T_h on the highways and proportional to the vehicle number and equal to ωn_c, $\omega > 0$, on the urban roads. Denoting as previously n_h and n_c the vehicle numbers on highway and in the city, the vehicle jump rate for the highway is

$$r_h(n_h) = \frac{n_h}{T_h}, \tag{54}$$

while it is

$$r_c(n_c) = \begin{cases} 1/\omega, & \text{if } n_c > 0 \\ 0, & \text{otherwise} \end{cases} \tag{55}$$

for the urban road. We measure the system behaviour in stationary states when the system performance is steady. We then calculate the average travel time and the number of vehicles taking the shortcut. Initially, all the vehicles are located at the origin. However, further simulation results show that the system is ergodic in the sense that it has a unique stationary distribution, regardless of the initial conditions.

We analyse and compare five routing strategies: A deterministic multi-class model for which each vehicle has fixed routing choice; A static random model for which the probability to choice each of the roads is fixed; Two dynamic assignment models for which the route is determined by minimising the travel time in real time, as a navigator may do; A minority game approach, where the route is determined according to previous travel times. The first two strategies rely on static assignment models, while the last three are based on dynamic assignment. The simulation model with static assignment is an attractive zero-range process with a product-form invariant distribution that can be determined recursively for a finite system [60, 61]. Alternatively, the assignment models can be addressed using Markovian queuing theory, as in [47–49]. In the following figures, we present Monte Carlo simulation results averaged on 10^6 jumps of the periodic system where the demand remains constant. The number of vehicles N ranges from 10 to 440 vehicles in steps of 10. We also use $T_h = 1$ while $\omega = 0.01$. The averaged travel times of the static assignment stochastic simulation models in Sec. 3.1 match directly the deterministic estimate of user equilibrium in stationary states. However, the dynamic assignment models in Sec. 3.2 show more complex behaviour that may reduce or exacerbate the Braess paradox.

3.1 Static assignment

In this section, we simulate two user optimum static assignment models. In the first one – the multi-class routing model – a constant route assignment is given to each of the vehicles. In the second one – the probabilistic routing model – the route is randomly determined at the origin according to a constant probability. Both simulation models match in average the performance of the deterministic system in stationary states.

3.1.1 Multi-class routing model

We consider in the multi-class model that each vehicle has a fixed routing strategy. This strategy is given by the deterministic user equilibrium m_* in (38). We show in Fig. 6 the mean simulation results of this multi-class routing model. As expected, the mean simulation results match the dynamic of the deterministic system in stationary states.

3.1.2 Probabilistic routing model

In the probabilistic routing model, the vehicles at the origin of the network randomly choose their route according to a fixed probability given by the user optimum (38) in stationary states. In this approach, the route choice probability has to be weighted by the

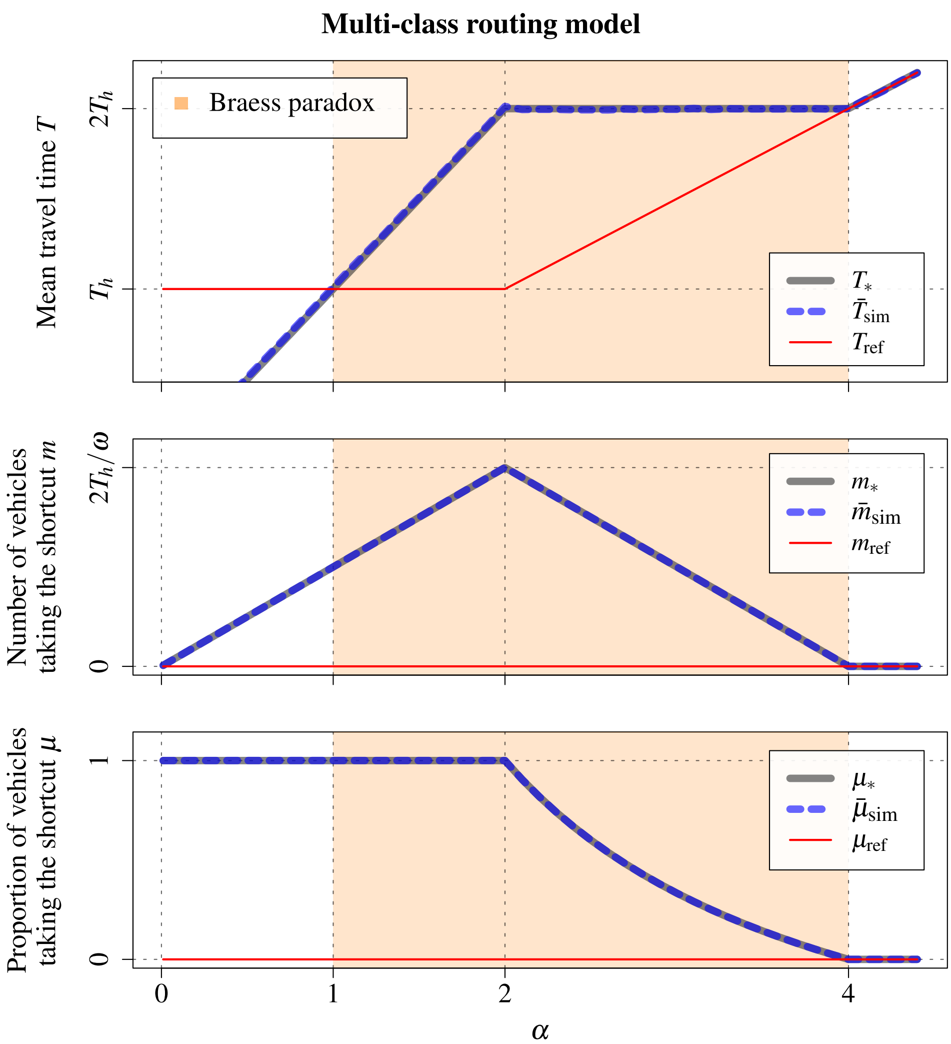

Figure 6 Mean travel time (upper panel) and number and proportion of drivers choosing the shortcut (lower panels) according to α for the user optimum in stationary states (38) – grey curve, the reference system without shortcut (34) – red curve, and mean simulation results of the stochastic interacting particle system with the multi-class routing model – blue dashed curve. The averages of the simulations are consistent with the deterministic results.

inverse of the travel time to avoid the bias induced by an over-representation of the faster vehicles. For the root *OABD* with the shortcut, this probability reads

$$p(k,m) = \frac{mT_{\frown}(k,m)}{(N-m)T_{-}(k,m) + mT_{\frown}(k,m)} \tag{56}$$

where k is the solution of the flow balancing equation (31). Then, the probability of choosing the route *OAD* or *OBD* without shortcut is given by

$$p_{\frown}(k,m) = \frac{1-p(k,m)}{2}. \tag{57}$$

Since the travel times over the different routes are equal for the user optimum, the routing probability reads directly, see (38),

$$p_* = p(k_*, m_*) = \frac{m_*}{N} = \begin{cases} 1, & 0 \le N < 2T_h/\omega, \\ \dfrac{4T_h}{\omega N} - 1, & 2T_h/\omega \le N < 4T_h/\omega, \\ 0, & N \ge 4T_h/\omega. \end{cases} \tag{58}$$

As for the multi-class routing model, the mean simulation results of the probabilistic route choice model are given by the deterministic system in stationary states (38), see Fig. 7. This is not surprising as both multi-class and probabilistic routing models are static strategies whose effects vanish on average.

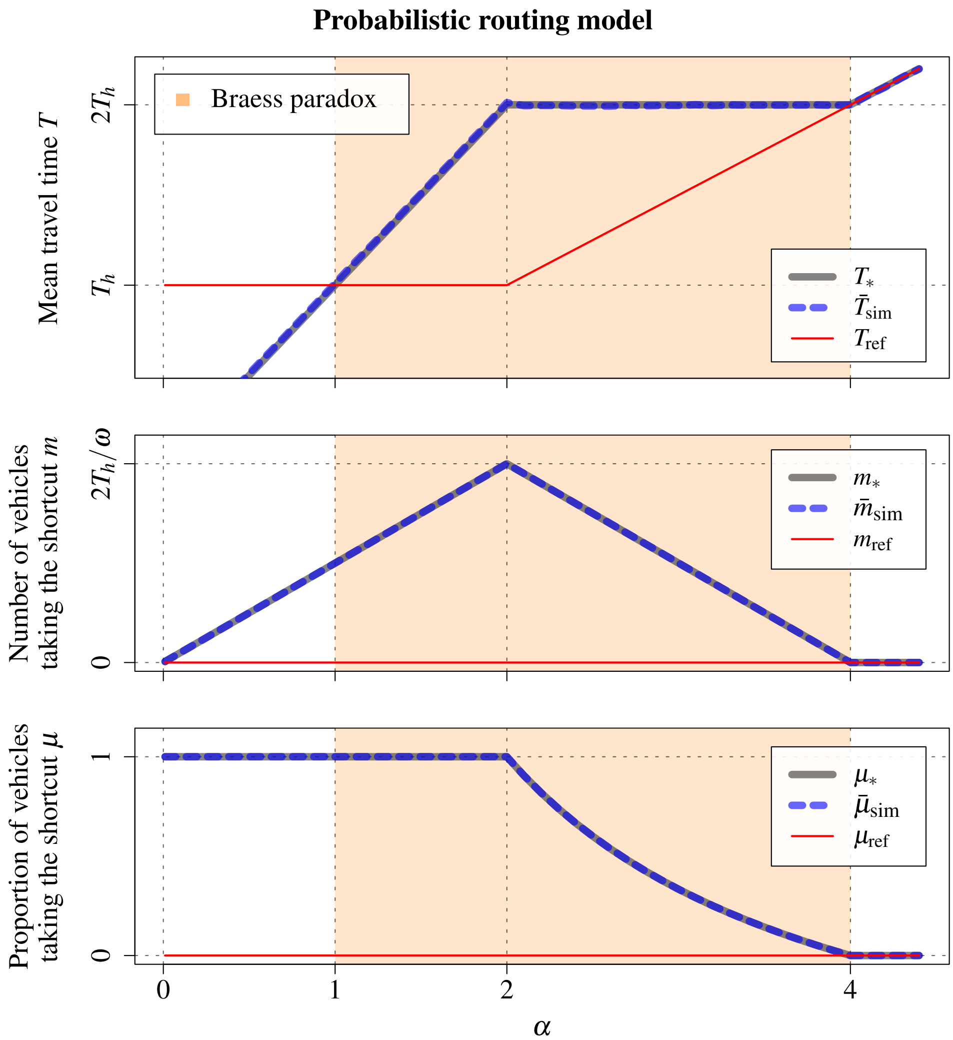

Figure 7 Mean travel time (upper panel) and number and proportion of drivers choosing the shortcut (lower panels) according to α for the user optimum in stationary states (38) – grey curve, the reference system without shortcut (34) – red curve, and mean simulation results of the stochastic interacting particle system with the fixed probabilistic routing model – blue dashed curve. The averages of the simulations are consistent with the deterministic model.

3.2 Dynamic assignment

In this section, we simulate three dynamic assignment models with individual route choice in real time. In the first one – the dynamic routing model – the route is determined at the origin based on the current mean travel time over the three routes. In the second – the full dynamic routing model – an update is made at the node A with the current mean travel times on the roads BD and AD. The last dynamic assignment model is a minority game where the driver chooses the route at the origin based on the last travel times. The simulation results for dynamic assignment differ from those for static assignment.

3.2.1 Dynamic routing model

We consider a state-dependent dynamic assignment model in which the driver at the origin O of the network selects the route that minimises the travel time in real time. Let $n_A(t)$ be the instantaneous number of vehicles on the urban road OA (respectively, $n_B(t)$ for the urban road BD), the dynamic routing strategy is

$$\mathrm{route}_O(t) = \arg\min\left\{\omega n_A(t) + T_h,\ \omega n_B(t) + T_h,\ \omega\big(n_A(t) + n_B(t)\big)\right\}. \tag{59}$$

for the vehicles located at the origin O[1]. Note that under certain conditions, the underlying Markov process of the system with state-dependent assignment is recurrent positive, i.e., ergodic and with a single and stable invariant distribution that can be determined analytically [48, 49]. In the following figure, we estimate the stationary expected value by means of Monte Carlo simulation. Interestingly, the mean simulation results of the dynamic routing strategy are not consistent with the deterministic results of the user optimum in stationary states (38), nor with the mean simulation results of the static routing models in Sec. 3.1 (compare Fig. 5, 6, 7 with Fig. 8). Dynamic assignment attenuates the Braess paradox at intermediate densities before overcoming it at high densities. In particular, the dynamic assignment affects the route, including the shortcut, in situations of extreme congestion involving a large number of vehicles, i.e. $N \geq 4T_h/\omega$, or equivalently, when $4 \leq \alpha$. Neither the deterministic user nor the system optimums does this.

3.2.2 Full dynamic routing model

In this section, we consider a state-dependent dynamic assignment model in which drivers choose the route that minimises travel time in real time not only at the origin O of the network, but also at the node A, as a navigator might do. In fact, at the node A, a driver can choose the motorway AD or the city road BD by using the shortcut AD (see Fig. 2). With $n_A(t)$ the current number of vehicles on the urban road OA (respectively, $n_B(t)$ for the urban road BD), the dynamic routing strategy is

$$\mathrm{route}_O(t) = \arg\min\left\{\omega n_A(t) + T_h,\ \omega n_B(t) + T_h,\ \omega\big(n_A(t) + n_B(t)\big)\right\}. \tag{60}$$

[1] An open-source online simulation platform of this model is available at https://www.vzu.uni-wuppertal.de/fileadmin/site/vzu/Simulating_Braess-Paradox.html?speed=0.9.

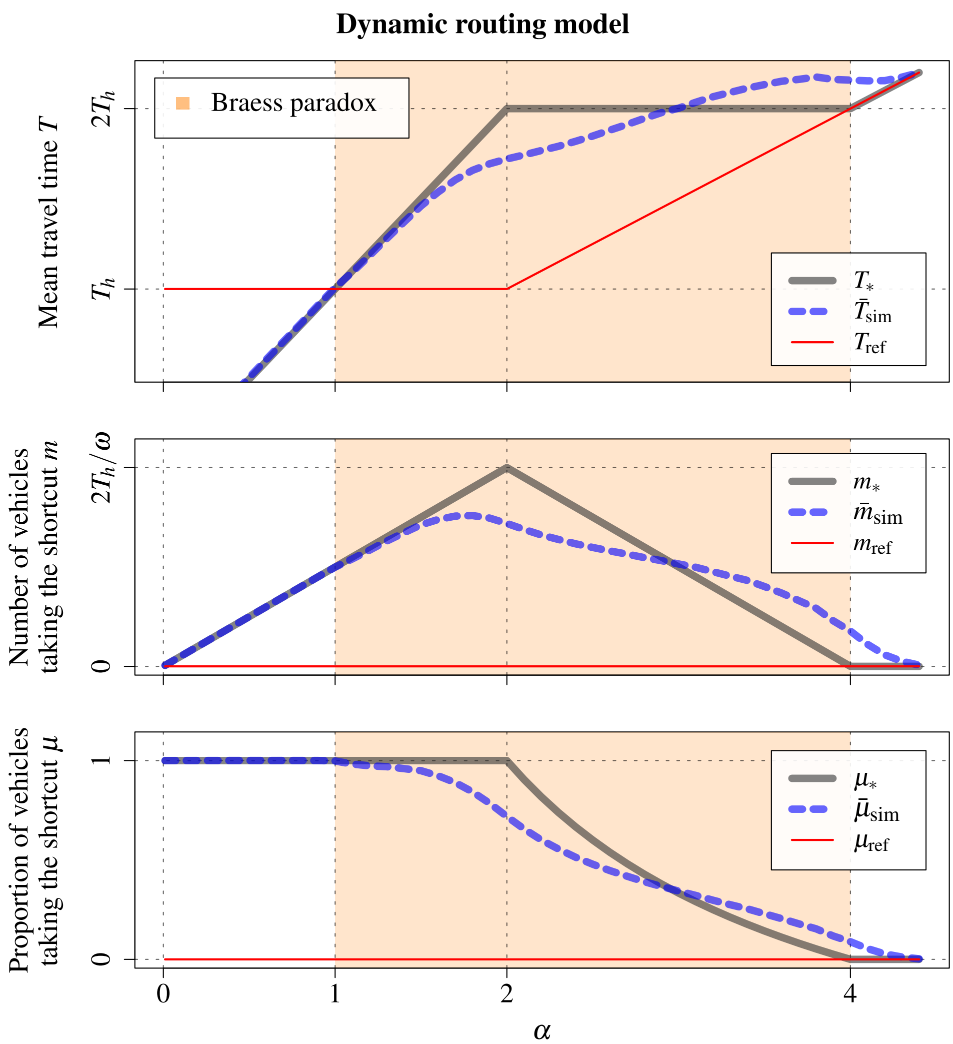

Figure 8 Mean travel time (upper panel) and number and proportion of drivers choosing the shortcut (lower panels) for the user optimum in stationary states (38) – grey curve, the reference system without shortcut (34) – red curve, and mean simulation results of the stochastic interacting particle system with the dynamic routing strategy (59) – blue dashed curve.

for the vehicles at origin O, and

$$\text{route}_A(t) = \arg\min\{T_h,\ \omega n_B(t)\}. \tag{61}$$

for the vehicles at node A. With this strategy, it turns out that the mean values almost match the estimates of the static user optimum in stationary states (see Fig. 9, top panel), although the shortcut is systematically underused (see Fig. 9, bottom panels). In fact, updating the route along the way significantly affects the results (compare Fig. 8 and 9).

3.2.3 Minority game

In contrast to previous dynamic traffic assignment models, in which routes are determined based on current traffic states, this model chooses routes based on past travel times, as in

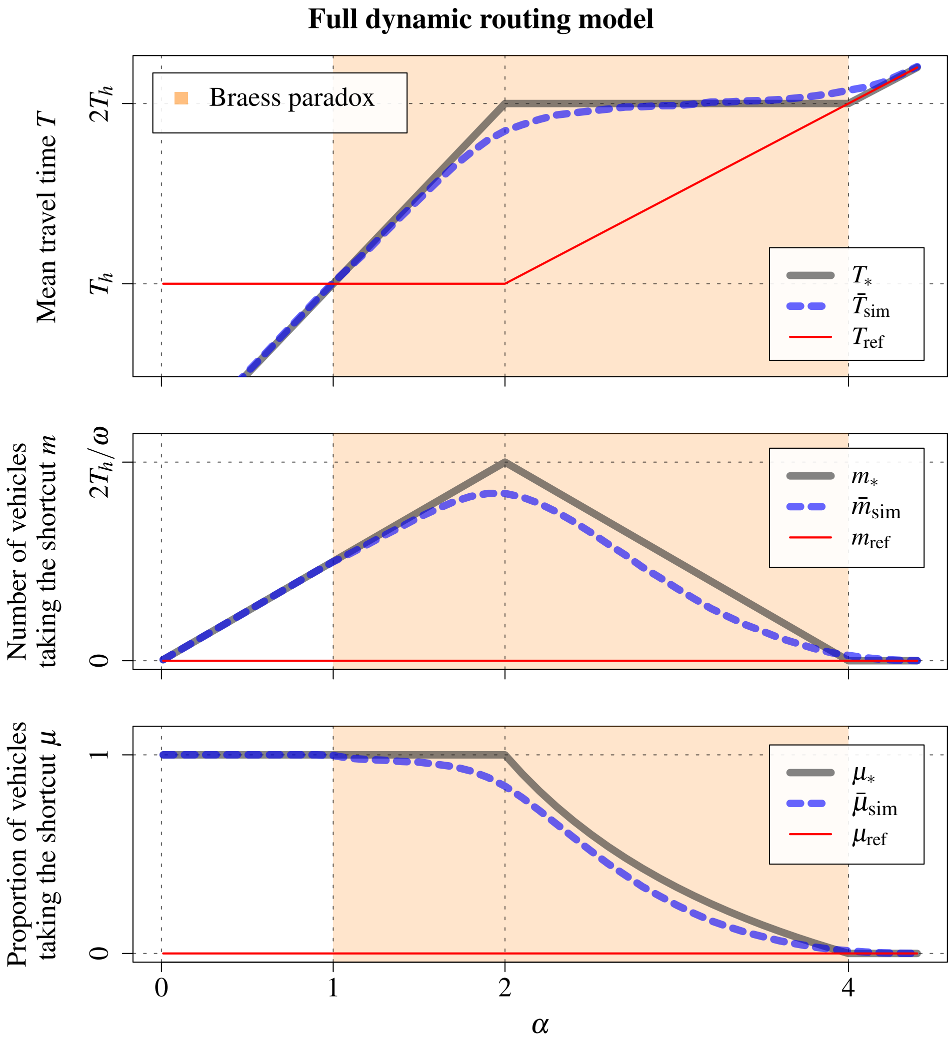

Figure 9 Mean travel time (upper panel) and number and proportion of drivers choosing the shortcut (lower panels) according to α for the user optimum in stationary states (38) – grey curve, the reference system without shortcut (34) – red curve, and mean simulation results of the stochastic interacting particle system with the full dynamic routing strategy (60) and (61) – blue dashed curve.

a minority game [62, 63]. Fig. 10 shows the simulation results for a stationary minority game, in which drivers choose routes that minimise the mean travel time over the last ten journeys on each route. Here, one hundred simulations over 5e4 jumps are repeated for a number of vehicles ranging from 10 to 440 in steps of 10. The simulation results show a much more pronounced discrepancy with the performance of the deterministic, static user optimum in stationary states given in (38), especially at high densities. In addition, the results exhibit a large variability. Further results with a memory of only two and five past travel times show similar results (compare Fig. 10 where the route is determined based on the ten past travel times and Fig. 11 where the route is determined based on the only two past travel times). Memory seems to be irrelevant in the minority game [64]. Note that other simulation experiments in which drivers statically choose a route based on their initial travel times show polarised behaviour, with almost all vehicles choosing the same route in stationary states.

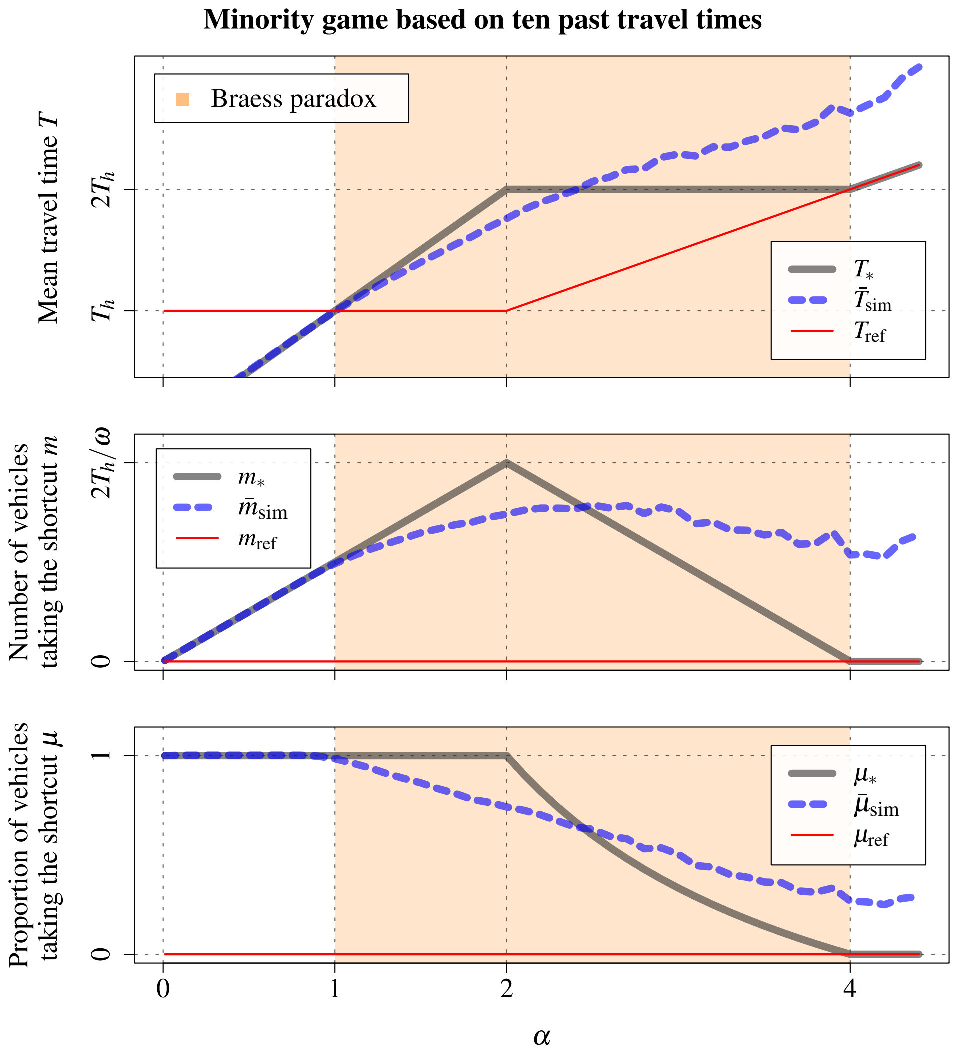

Figure 10 Mean travel time (upper panel) and number and proportion of drivers choosing the shortcut (lower panels) according to α for the user optimum in stationary states (38) – grey curve, the reference system without shortcut (34) – red curve, and mean simulation results of the stochastic interacting particle system with the minority game where the drivers choose the route based on their ten past travel times – blue dashed curve.

4 Conclusion

In this article, we present explicit results for the Braess paradox in a closed system. The parameters are the (constant) number of vehicles in the system $N \geq 1$, the travel time on the highway $T_h > 0$ and the headway coefficient $\omega > 0$ for congested urban roads.

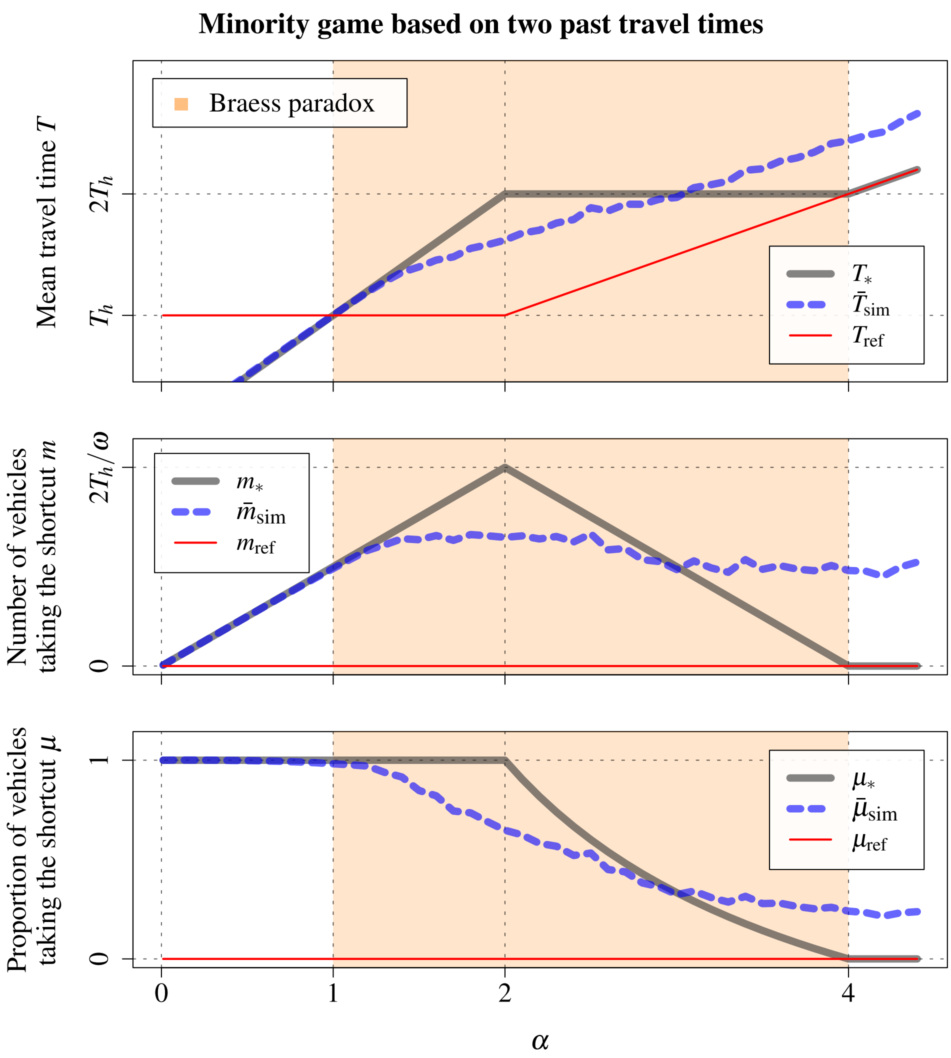

Figure 11 Mean travel time (upper panel) and number and proportion of drivers choosing the shortcut (lower panels) according to α for the user optimum in stationary states (38) – grey curve, the reference system without shortcut (34) – red curve, and mean simulation results of the stochastic interacting particle system with the minority game where the drivers choose the route based on their two past travel times – blue dashed curve.

We consider first the deterministic system in stationary states with flow balancing in the user and system optimum. Monte Carlo simulations of a stochastic interacting particle system with static routing strategy match the deterministic stationary performance. The Braess paradox occurs for the density range $T_h/\omega \leq N \leq 4T_h/\omega$, i.e., $1 \leq \alpha \leq 4$ and its effect is maximal for $N = 2T_h/\omega$, i.e. $\alpha = 2$. However, dynamic assignments, where the driver chooses the route that minimises the travel time in real time according to the current state of the network or minority game where the drivers choose the route based on their past travel times, show a deviation from the theoretical results. The Braess paradox first diminishes at intermediate densities before being exacerbated at high densities. The results highlight that individual dynamic assignment strategies can lead to unexpected behaviours. They point out the necessity to develop collective strategies and further control techniques to limit and control the occurrence of the Braess paradox in traffic networks.

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Author Contributions SL: Conceptualization, Methodology, Investigation, Writing – review & editing. AS: Conceptualization, Methodology, Investigation, Writing – review & editing. AT: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing.

Reproducibility of the results An open source simulation platform of the Braess paradox with the stochastic interacting particle system with the dynamic routing model given in (59) is available online at

https://www.vzu.uni-wuppertal.de/fileadmin/site/vzu/Simulating_Braess-Paradox.html?speed=0.9

Conflict of interest The authors declare that they have no conflict of interest.

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Appendix: Uniform state

In Sec. 2, we compute the system performance in stationary states with flow balancing. We consider here a uniform state where the vehicles are equally distributed between urban roads and highways. For example, if n vehicles are using the urban road and the highway, we assume that $n/2$ vehicles are on the highway and $n/2$ on the urban road (see Fig. A1). Similarly if m vehicles are using the shortcut, there will be $m/2$ of these vehicles on each of the urban roads. The uniform state is not necessary stationary except for the user equilibrium of the system including the shortcut up to a certain density threshold.

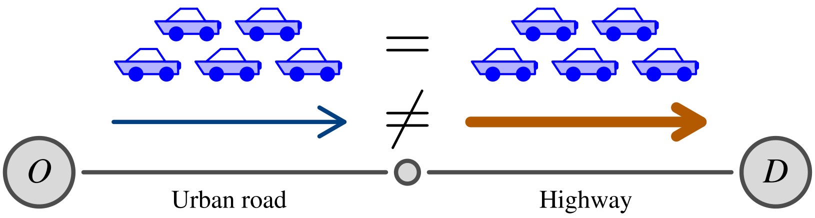

Figure A1 Scheme for the uniform state where the number of vehicles are equally distributed between the urban road and highway. As the speeds on the two roads may be different, the flows are not necessary equal (i.e., the state is not necessary stationary).

Proposition A1 (Mean travel time in uniform states)**.** We have in uniform states where the vehicles are equally distributed between the roads

$$T^u_{\smile}(m) = \omega\frac{N+m}{4} + T_h, \qquad m \in [0,N], \tag{A1}$$

$$T^u_{-}(m) = \omega\frac{N+m}{2}, \qquad m \in [0,N], \tag{A2}$$

and

$$T^u(m) = \omega\frac{(N+m)^2}{4N} + T_h\frac{N-m}{N} \qquad m \in [0,N], \tag{A3}$$

or again, with the proportion of vehicle using the shortcut and relative travel times

$$\mathcal{T}^u_{\smile}(\mu) = 1 + \frac{\alpha}{4}(1+\mu), \qquad \mu \in [0,1], \tag{A4}$$

$$\mathscr{T}_{-}^{u}(\mu) = \frac{\alpha}{2}(1+\mu), \qquad \mu \in [0,1], \tag{A5}$$

and

$$\mathscr{T}^{u}(\mu) = \frac{\alpha}{4}(1+\mu)^2 + 1 - \mu, \qquad \mu \in [0,1]. \tag{A6}$$

Proof. It is straightforward to check that in uniform states where the vehicles are equally distributed between urban roads and highways we have

$$\begin{aligned} n_c^u(m) &= \frac{1}{2}(n+m) \\ &= \frac{1}{4}(N+m), \qquad m \in [0,N], \end{aligned} \tag{A7}$$

as $n = (N-m)/2$, see (1).

Then using (3), (4), and (5), it follows that

$$\begin{aligned} T_{\sim}^{u}(m) &= \omega n_c^u(m) + T_h \\ &= \omega \frac{N+m}{4} + T_h, \qquad m \in [0,N], \end{aligned} \tag{A8}$$

$$\begin{aligned} T_{-}^{u}(m) &= 2\omega n_c^u(m) \\ &= \omega \frac{N+m}{2}, \qquad m \in [0,N], \end{aligned} \tag{A9}$$

and

$$\begin{aligned} T^{u}(m) &= \omega n_c(m) \frac{N+m}{N} + T_h \frac{N-m}{N} \\ &= \omega \frac{(N+m)^2}{4N} + T_h \frac{N-m}{N}, \qquad m \in [0,N]. \end{aligned} \tag{A10}$$

Finally the relative travel times (A4), (A5) and (A6) are recovered by using $\mathscr{T} = T/T_h$, $\mu = m/N$ and $\alpha = N\omega/T_h$. $\square$

Remark A1. *We have on the highways that*

$$\begin{aligned} n_h^u(m) &= \frac{1}{2}n \\ &= \frac{1}{4}(N-m), \qquad m \in [0,N], \end{aligned} \tag{A11}$$

and the total vehicle number in the system is conserved by construction

$$2(n_c^u(m) + n_h^u(m)) = 2\left(\frac{1}{4}(N+m) + \frac{1}{4}(N-m)\right) = N \tag{A12}$$

Reference system without shortcut

Proposition A2 (Mean travel time in uniform states without shortcut)**.** Without shortcut (i.e., $m = 0$), the reference travel time in uniform states is

$$T^u_{\text{ref}} = \frac{1}{4}\omega N + T_h, \tag{A13}$$

while the relative reference travel time is

$$\mathscr{T}^u_{\text{ref}} = 1 + \frac{\alpha}{4}. \tag{A14}$$

with $\alpha = N\omega/T_h$.

Proof. We have directly using (A3)

$$T^u_{\text{ref}} = T^u(0) = \frac{1}{4}\omega N + T_h. \tag{A15}$$

or again

$$\mathscr{T}^u_{\text{ref}} = \frac{T^u_{\text{ref}}}{T_h} = 1 + \frac{\alpha}{4}, \qquad \alpha = N\omega/T_h. \tag{A16}$$

□

User optimum

Proposition A3 (User optimum in uniform states)**.** The user optimum in uniform states where the vehicles are equally distributed between the roads is given by

$$m^u_* = \begin{cases} N, \\ 4T_h/\omega - N, \\ 0, \end{cases} \qquad T^u_* = \begin{cases} \omega N, & 0 \le N < 2T_h/\omega, \\ 2T_h, & 2T_h/\omega \le N < 4T_h/\omega, \\ \omega N/4 + T_h, & N \ge 4T_h/\omega, \end{cases} \tag{A17}$$

or again, with the proportion of vehicle using the shortcut and relative travel time,

$$\mu^u_* = \begin{cases} 1, \\ 4/\alpha - 1, \\ 0, \end{cases} \qquad \mathscr{T}^u_* = \begin{cases} \alpha, & 0 \le \alpha < 2, \\ 2, & 2 \le \alpha < 4, \\ 1 + \alpha/4, & \alpha \ge 4. \end{cases} \tag{A18}$$

Proof. In the user equilibrium, we are looking for a number of drivers $\tilde{m}^u_*$ choosing the route *OABD* with the shortcut where all travel times are equal, i.e., for which

$$T^u_{\frown}(\tilde{m}^u_*) = T^u_{-}(\tilde{m}^u_*). \tag{A19}$$

Using (A1) and (A2), this is

$$T_h + \omega\frac{N + \tilde{m}^u_*}{4} = \omega\frac{N + \tilde{m}^u_*}{2}, \tag{A20}$$

which gives

$$\tilde{m}_*^u = 4T_h/\omega - N. \tag{A21}$$

In addition, the number of drivers m_*^u choosing the shortcut is necessary positive and lower than N, i.e.,

$$m_*^u = \min\{N, \max\{0, \tilde{m}_*^u\}\}. \tag{A22}$$

This gives

$$m_*^u = \begin{cases} N, & 0 \le N < 2T_h/\omega, \\ 4T_h/\omega - N, & 2T_h/\omega \le N < 4T_h/\omega, \\ 0 & N \ge 4T_h/\omega. \end{cases} \tag{A23}$$

Then, using (A3) we obtain the corresponding mean travel time of the user equilibrium in uniform states

$$T_*^u = T^u(m_*^u) = \begin{cases} \omega N, & 0 \le N < 2T_h/\omega, \\ 2T_h, & 2T_h/\omega \le N < 4T_h/\omega, \\ \omega N/4 + T_h, & N \ge 4T_h/\omega. \end{cases} \tag{A24}$$

Finally the proportion of vehicles using the shortcut and relative travel times (A18) are recovered by using $\mathscr{T} = T/T_h$, $\mu = m/N$ and $\alpha = N\omega/T_h$. □

Remark A2. *The shortcut is used by all the vehicles up to $\alpha \le 2$, i.e. $N \le 2T_h/\omega$, and remains in use up to $\alpha \le 4$, i.e. $N \le 4T_h/\omega$. After this threshold, the travel time on the route OABD is longer than those on the routes OAD and OBD including the highway, even for a single vehicle.*

Remark A3. *The travel time difference to the reference system with no shortcut*

$$\Delta T = T - T_{ref} \tag{A25}$$

quantifies the occurrence of the Braess paradox. It is positive (resp. negative) when the Braess paradox holds (resp. does not hold). We have in uniform states for the user optimum

$$\Delta T_*^u = T_*^u - T_{ref}^u = \begin{cases} \frac{3}{4}\omega N - T_h \le 0, & 0 \le N < \frac{4}{3}T_h/\omega, \\ \frac{3}{4}\omega N - T_h \ge 0, & \frac{4}{3}T_h/\omega \le N < 2T_h/\omega, \\ T_h - \omega N/4 \ge 0, & 2T_h/\omega \le N < 4T_h/\omega, \\ 0, & N \ge 4T_h/\omega. \end{cases} \tag{A26}$$

Therefore, the Braess paradox arises for the density range

$$\frac{4}{3}T_h/\omega \le N \le 4T_h/\omega. \tag{A27}$$

or again

$$\frac{4}{3} \le \alpha \le 4. \tag{A28}$$

with $\alpha = N\omega/T_h$.

System optimum

Proposition A4 (System optimum in uniform states)**.** The system optimum in uniform states where the vehicles are equally distributed between the roads is given by

$$m^u_{\text{opt}} = \begin{cases} N, \\ 2T_h/\omega - N, \\ 0, \end{cases} \qquad T^u_{\text{opt}} = \begin{cases} \omega N, & 0 \leq N < T_h/\omega, \\ 2T_h - \frac{1}{\omega N} T_h^2, & T_h/\omega \leq N < 2T_h/\omega, \\ \omega N/4 + T_h, & N \geq 2T_h/\omega, \end{cases} \tag{A29}$$

or again, with the proportion of vehicle using the shortcut and relative travel time,

$$\mu^u_{\text{opt}} = \begin{cases} 1, \\ 2/\alpha - 1, \\ 0, \end{cases} \qquad \mathscr{T}^u_{\text{opt}} = \begin{cases} \alpha, & 0 \leq \alpha < 1, \\ 2 - 1/\alpha, & 1 \leq \alpha < 2, \\ 1 + \alpha/4, & \alpha \geq 2. \end{cases} \tag{A30}$$

Proof. The system optimum is obtained by minimising the mean travel time (A3). We have

$$(T^u)'(m) = \omega \frac{N+m}{2N} - \frac{T_h}{N}, \qquad m \in [0, N]. \tag{A31}$$

Then, $(T^u)''(m) = \omega/(2N) \geq 0$ and the mean travel time is minimal for m^u_{opt} such that $(T^u)'(m^u_{\text{opt}}) = 0$. This gives

$$\tilde{m}^u_{\text{opt}} = \frac{2T_h}{\omega} - N. \tag{A32}$$

Bounding the number of drivers choosing the shortcut in $[0, N]$, i.e.,

$$m^u_{\text{opt}} = \min\{N, \max\{0, \tilde{m}^u_{\text{opt}}\}\}, \tag{A33}$$

we obtain

$$m^u_{\text{opt}} = \begin{cases} N, & 0 \leq N < T_h/\omega, \\ 2T_h/\omega - N, & T_h/\omega \leq N < 2T_h/\omega, \\ 0, & N \geq 2T_h/\omega. \end{cases} \tag{A34}$$

Then, using (A3) we obtain the mean travel time of the system optimum in uniform state

$$T^u_{\text{opt}} = T^u(m^u_{\text{opt}}) \begin{cases} \omega N, & 0 \leq N < T_h/\omega, \\ 2T_h - \frac{1}{\omega N} T_h^2, & T_h/\omega \leq N < 2T_h/\omega, \\ \omega N/4 + T_h, & N \geq 2T_h/\omega. \end{cases} \tag{A35}$$

Finally the proportion of vehicles using the shortcut and relative travel times (A30) are recovered by using $\mathscr{T} = T/T_h$, $\mu = m/N$ and $\alpha = N\omega/T_h$. □

Remark A4. *For the system optimum, in contrast to the user optimum, the shortcut is exclusively in use for $\alpha \leq 1$, i.e. $N \leq T_h/\omega$, partially in use up to $\alpha \leq 2$, i.e. $N \leq 2T_h/\omega$, and no more in use for $\alpha \geq 2$, i.e. $N \geq 2T_h/\omega$.*

Remark A5. *In uniform states, the travel time difference to the reference system with no shortcut and the system optimum*

$$\Delta T_{opt}^{u} = T_{opt}^{u} - T_{ref}^{u} = \begin{cases} \frac{3}{4}\omega N - T_h \leq 0, & 0 \leq N < T_h/\omega, \\ T_h - \frac{1}{\omega N} T_h^2 - \omega N/4 \leq 0, & T_h/\omega \leq N < 2T_h/\omega, \\ 0, & N \geq 2T_h/\omega. \end{cases} \tag{A36}$$

is always negative. Indeed, the Braess paradox only arises for the user optimum.

The mean travel time and number of vehicles taking the shortcut for systems in uniform states are shown in Fig. A2. The continuous grey curve is the user optimum, the dashed blue curve is the system optimum, and the red curve is the reference system with no shortcut. The Braess paradox for the user optimum is shown in the orange areas. It occurs for the density range $\frac{4}{3} \leq \alpha \leq 4$, i.e. $\frac{4}{3}T_h/\omega \leq N \leq 4T_h/\omega$ and is maximal for $\alpha = 2$, i.e. $N = 2T_h/\omega$. In particular, it does not occur at high densities (i.e., $\alpha \geq 4$ or $N \geq 4T_h/\omega$), where the user and system optimums imply a systematic use of highways (see also [65]). Note that the system optimum makes marginal use of the shortcut, with performance improvement at low densities only (i.e., $\alpha \leq 2$ or $N \leq 2T_h/\omega$).

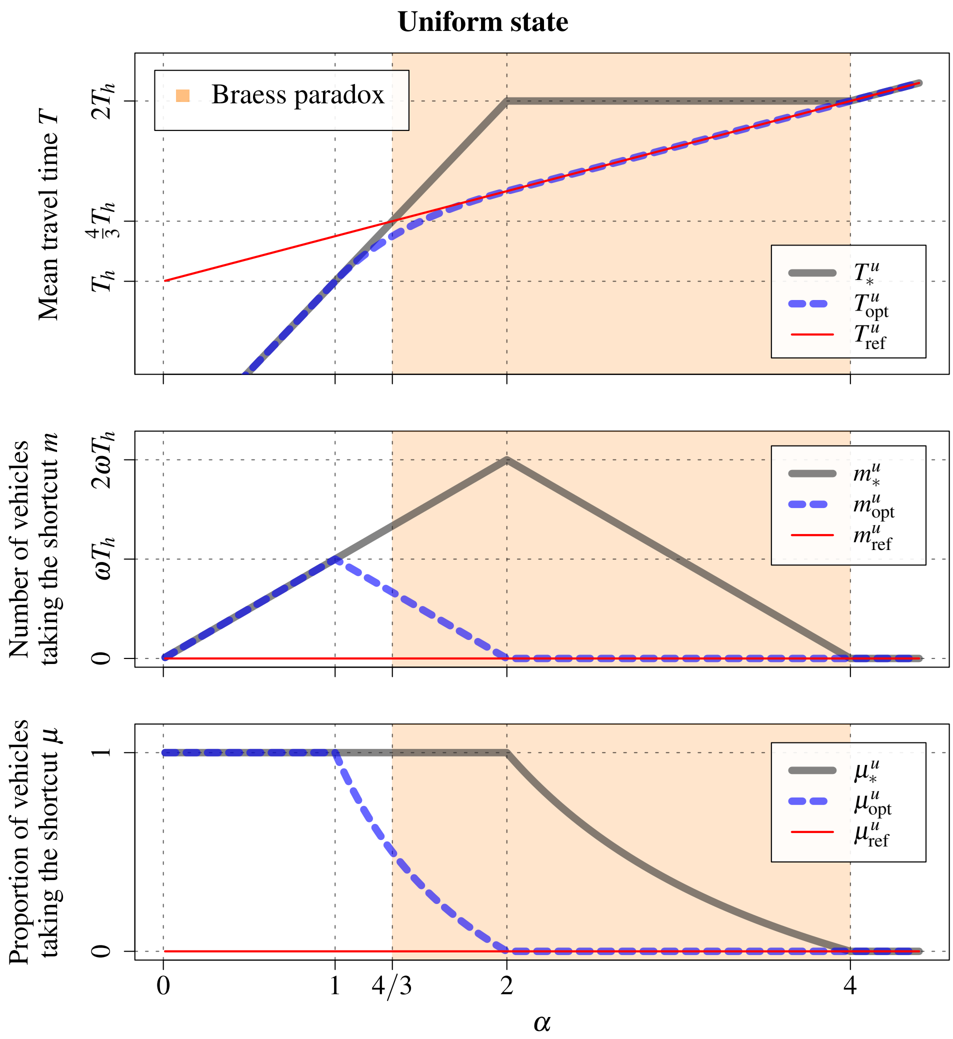

Figure A2 Mean travel time (upper panel) and number and proportion of drivers choosing the shortcut (lower panels) in uniform states according to $\alpha = N\omega/T_h$ for the user optimum (A17) and (A18) – grey curve, the system optimum (A29) and (A30) – blue dashed curve, and the reference system without the shortcut (A13) – red curve. The orange areas indicate the density range for which the Braess paradox occurs, i.e., where the user optimum travel time with the shortcut is greater than without the shortcut.